

2D DILATON GRAVITY MADE COMPACT

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Abstract

We show that the equations of motion of two-dimensional dilaton gravity conformally coupled to a scalar field can be reduced to a single non-linear second-order partial differential equation when the coordinates are chosen to coincide with the two scalar fields, the matter field f and the dilaton ϕ , which are present in the theory. This result may help solve and understand two- and higher-dimensional classical and quantum gravity.

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Low-dimensional models of gravity are receiving a great deal of attention lately as they can provide insight into the classical as well as the quantum theories of more realistic (higher-dimensional) theories of gravity. Prominent among these low-dimensional general covariant dynamical systems are the 2D dilaton models of gravity, the general action of which can be written in the form

$$S_{GDG} = S_V - S_M \quad (1)$$

where

$$S_V = \int d^2x \sqrt{-g} (R\phi + V(\phi)) \quad (2)$$

and S_M is a gravity-matter interaction term which may be more general but we shall take in the present letter to be of the form

$$S_M = \frac{1}{2} \int d^2x \sqrt{-g} \Omega(\nabla f)^2 \quad (3)$$

with Ω and unspecified function of the dilaton field ϕ , $\Omega = \Omega(\phi)$. A direct connection can be made with physical reality by noting that, for instance, spherically symmetric Einstein-Hilbert gravity minimally coupled to a massless scalar field coincides with the model with $V = 2/\sqrt{\phi}$ and $\Omega = G\phi$, with G the Newton constant.

Unfortunately, and notwithstanding their relative simplicity, most of these models, in particular spherically symmetric Einstein gravity, have eluded their being analitically solved at the classical level, let alone the quantum one. Actually, solving these model is relatively easy when no matter is present, as they are highly symmetric [1]. However, the introduction of matter fields breaks a great deal of these symmetries (although some are preserved) and makes solving them a much more difficult task [1, 2]. In fact, when coupled to conformal matter, they can be solved only for $V(\phi) = 4\lambda^2 e^{\beta\phi}$ with constant λ and β (the string-inspired and the exponential models) [3, 4] and for $V(\phi) = 4\lambda^2 \phi$ (the Jackiw-Teitelboim model) [5].

For arbitrary V , these models have been solved only for chiral matter. In this case, the (generalized Vaidya) solution is given by [1]

$$ds^2 = 2d\phi du + (2M(u) - J(\phi))(du)^2 \quad (4)$$

where $dJ/d\phi = V$ and

$$M(u) = \int^u d\tilde{u} T_{uu}(\tilde{u}) \quad (5)$$

with T_{uu} the only non-null component of the energy-momentum tensor.

Beyond these very particular cases, we have to resort, even at the classical level, to approximate solutions, such as the ones provided by numerical methods (see, for instance Ref. [6] and references therein), perturbative methods (see, for instance Ref. [7] and references therein) and so on. It is clear that this situation precludes our gaining full benefit from these models in seeking insight into quantum gravity. This obstruction is even more grave because of the fact that some of the problems we must overcome before arriving at a quantum theory of gravity can be traced back to our failure to reach a full understanding of the classical theory. The problem of time, for instance, already exists at the classical level, even though in this case it can be swept under the carpet [8]. Also, how and what we can observe remains largely a mystery even in the classical theory [9]. In general, there is a good case for arguing that the current approach to observations in the classical theory of gravity is, at the very least, rudimentary, and perhaps badly conceived and altogether inadequate to be extrapolated to the quantum theory. For instance, in any theory which aspires to be truly fundamental, the space-time manifold cannot be taken as a given primary concept but rather it should be derived from more basic principles.

The analysis in the present letter may help solve (and improve our understanding) of the classical two- and higher-dimensional models of gravity and help devise the quantum theories.

The Euler-Lagrange equations of motion of the models in Eq. (1) can be brought to the form:

$$R + V'(\phi) = T_\phi \quad (6)$$

$$\nabla_\mu \nabla_\nu \phi = \frac{1}{2} g_{\mu\nu} V - T_{\mu\nu} \quad (7)$$

$$\nabla_\mu (\Omega \nabla^\mu f) = 0 \quad (8)$$

where

$$T_\phi = \frac{\Omega'(\phi)}{2}(\nabla f)^2 \quad (9)$$

and

$$T_{\mu\nu} = \frac{\Omega}{2} \left\{ \nabla_\mu f \nabla_\nu f - \frac{1}{2} g_{\mu\nu} (\nabla f)^2 \right\} \quad (10)$$

As shown in Ref. [1], Eq. (6) can be deduced from Eq. (7) by using basic properties of the covariant derivatives ∇_μ and the curvature tensors in two dimensions. Therefore, we can concentrate on Eq. (7) and Eq. (8).

For chiral matter, an explicit compact solution, the generalized Vaidya solution (4), can be given in a gauge in which one of the coordinates has been made to coincide with the dilaton field. We propose going a step further and taking a gauge such that each of the two coordinates coincides with each of the scalar fields, ϕ and f , in the theory. In other words, we will use ϕ and f as coordinates.

We are not going to dwell here on the quality of these coordinates, as they are, firstly and above all, a tool to help solve a certain system of equations. Therefore we are not going to analyze in this letter, for instance, how much of the spacetime can be covered with this system of coordinates. This and related questions will be considered in future communications. Nonetheless, these coordinates are quite natural – hence the chosen name – and incorporate much of what has been said in the literature about the necessity of using an internal time in gravity (see, for instance, Ref. [8, 9] and references therein). Our approach goes even further and uses not only an internal time, but an internal space-time. The points of the space-time are described by physical quantities, f and ϕ , which are subject to dynamics. In fact, this approach may be regarded as a modelling (in two dimensions) of relationism.

On the other hand, the technical advantages of these coordinates are apparent. Firstly, they render the equations in (7) first order. Secondly, the arbitrary functions which appear, V and Ω , are functions of the coordinates, which should facilitate a generic treatment of all the models. Thirdly –and this is an unexpected fact, the origin of which remains hidden to us–, in these coordinates it turns out that the

equation of motion for the matter field, Eq. (8), follows from Eq. (7). Therefore all the equations of motion are brought into a system of three first-order partial differential equations with two independent variables and three dependent variables (the component of the metric tensor).

This system, if expressed in contravariant form

$$\nabla^\mu \nabla^\nu \phi = \frac{1}{2} g^{\mu\nu} V - \frac{\Omega}{2} \left\{ \nabla^\mu f \nabla^\nu f - \frac{1}{2} g^{\mu\nu} (\nabla f)^2 \right\} \quad (11)$$

can be written in the form

$$-\Gamma^{\phi, \mu\nu} = \frac{1}{2} g^{\mu\nu} V - \frac{\Omega}{2} \left\{ \nabla^\mu f \nabla^\nu f - \frac{1}{2} g^{\mu\nu} (\nabla f)^2 \right\} \quad (12)$$

where $\Gamma^{\lambda, \mu\nu}$, the “contravariant” Christoffel symbols, involve the contravariant metric tensor only:

$$\begin{aligned} \Gamma^{\lambda, \mu\nu} &\equiv g^{\alpha\mu} g^{\beta\nu} \Gamma_{\alpha\beta}^\lambda \\ &= -\frac{1}{2} \left(g^{\mu\rho} \partial_\rho g^{\lambda\nu} + g^{\nu\rho} \partial_\rho g^{\lambda\mu} - g^{\lambda\rho} \partial_\rho g^{\mu\nu} \right) \end{aligned} \quad (13)$$

It is clear that the system in Eq. (12) will take the simplest form if the three dependent variables are chosen to coincide with the components of the contravariant metric tensor $g^{\mu\nu}$.

If we make $g^{\phi\phi} = F$, $g^{\phi f} = g^{f\phi} = G$ and $g^{ff} = H$, we have

$$\Gamma^{\phi, \phi\phi} = -\frac{1}{2} F \partial_\phi F - \frac{1}{2} G \partial_f F \quad (14)$$

$$\Gamma^{\phi, \phi f} = -\frac{1}{2} H \partial_f F - \frac{1}{2} G \partial_\phi F \quad (15)$$

$$\Gamma^{\phi, ff} = -G \partial_\phi G - H \partial_f G + \frac{1}{2} F \partial_\phi H + \frac{1}{2} G \partial_f H \quad (16)$$

and

$$T^{\phi\phi} = \frac{\Omega}{2} \left(G^2 - \frac{1}{2} F H \right) \quad (17)$$

$$T^{\phi f} = \frac{\Omega}{4} G H \quad (18)$$

$$T^{ff} = \frac{\Omega}{4} H^2 \quad (19)$$

Therefore, the equations of motion take the form

$$F\partial_\phi F + G\partial_f F = FV - \Omega \left(G^2 - \frac{1}{2}FH \right) \quad (20)$$

$$H\partial_f F + G\partial_\phi F = GV - \frac{\Omega}{2}GH \quad (21)$$

$$G\partial_\phi G + H\partial_f G - \frac{1}{2}F\partial_\phi H - \frac{1}{2}G\partial_f H = \frac{1}{2}HV - \frac{\Omega}{4}H^2 \quad (22)$$

Now, a bit of algebra with the first and second equations yields

$$\left. \begin{aligned} \partial_f F &= -\Omega G \\ \partial_\phi F &= V + \frac{\Omega}{2}H \end{aligned} \right\} \Rightarrow -\partial_\phi(\Omega G) = \frac{\Omega}{2}\partial_f H \quad (23)$$

and

$$H\partial_f G - \frac{1}{2}F\partial_\phi H - G\partial_f H = \frac{1}{2}HV - \frac{\Omega}{4}H^2 + \frac{\partial_\phi \Omega}{\Omega}G^2 \quad (24)$$

Hence, all the complexity of these models have been encapsulated in Eq. (24), which in term of the single unknown function $D = F - J$ can be written (with obvious notation) in the form:

$$-2D_\phi D_{ff} + 2D_f D_{f\phi} + (D+J)(\Omega_\phi D_\phi - \Omega D_{\phi\phi}) - \Omega V D_\phi + \Omega(D_\phi)^2 - \frac{\Omega_\phi}{\Omega}(D_f)^2 = 0 \quad (25)$$

In summary, we have shown that in these coordinates the (contravariant) metric tensor $(g^{\mu\nu})$ can be expressed in term of a single function D :

$$(g^{\mu\nu}) = \begin{pmatrix} D+J & -\frac{1}{\Omega}D_f \\ -\frac{1}{\Omega}D_f & \frac{2}{\Omega}D_\phi \end{pmatrix} \quad (26)$$

This function D is not arbitrary but has to obey Eq. (25), which, therefore, encapsulates all the dynamical content of these models. We hope that the compactness and (relative) simplicity of this result will be useful to solve and understand two- and higher-dimensional classical and quantum gravity.

To finish, let us show with an example how this equation may actually serve to find solutions and perhaps the general trajectories of these models. If $\Omega = 1$

(minimally coupled matter), then $D_\phi = 0$ is a solution to Eq. (25). Eq. (23) implies then

$$(g^{\mu\nu}) = \begin{pmatrix} J + K & -\partial_f K \\ -\partial_f K & 0 \end{pmatrix} \quad (27)$$

with K an arbitrary function of f , $K = K(f)$. Clearly, this is the Vaidya solution, which can be checked with the corresponding change of coordinates.

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